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A Computational Fluid Dynamic Technique Valid at the Centerline for Non-Axisymmetric Problems in Cylindrical Coordinates

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A technique is described for the numerical solution of non-axisymmetric flow problems posed in cylindrical coordinates when the z-axis is included in the flowfield. The highlight of the technique is the manner in which the singularities at the centerline are handled. Specifically, the governing flowfield equations at $r = 0$ are put in a special form by applying L'Hospital's Rule. The required radial derivatives are evaluated using a one-sided, second-order accurate, first-difference. This leads to a smooth, convergent calculation of the flowfield at the centerline. This appears to be the first generally applicable numerical method for avoiding coordinate system singularites in the context of a finite-difference scheme, and could have application to many non-axisymmetric flows. The technique is illustrated by specific results for the time-dependent flowfield inside an internal combustion engine.

1. INTRODUCTION

The computational technique described here was developed in the course of efforts by the present authors [1, 2, 3] and R. Diwakar [3, 4, 5] to obtain numerical solutions for the flowfield in an internal combustion reciprocating engine. The work of References [2] and [3] was in part concerned with obtaining solutions for the particular three-dimensional model of Figure 1, which shows a model of the valve and cylinder arrangement used in a spark-ignition internal combustion engine. This geometrical model is most conveniently represented in cylindrical coordinates, as shown. Note that points on the z-axis are part of the flowfield and hence must be computed by whatever numerical algorithm is used. However, the z-axis is a singular line in the cylindrical coordinate system, because the value of the azimuthal coordinate ϕ is undefined along this line. Thus any flowfield variable which is a function of the coordinates (r, ϕ, z) will be mathematically undefined along the centerline, and may in fact be multivalued. The singularity at the centerline is easily recognized because the terms in the governing fluid mechanical equations containing $(1/r)$ are undefined at $r = 0$. For axisymmetric flow, symmetry conditions at the centerline allow a straightforward calculation on the z-axis. Such is not the case for non-axisymmetic flows. This paper describes what is thought to be the first generally applicable method

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for avoiding the coordinate system singularities associated with the numerical solution of non-axisymmetric problems in cylindrical coordinates.

Emphasis in the present paper focuses on the technique of solution, which is quite general. The results specifically obtained for the I. C. engine application are presented in References [2] and [3]. A cursory description of the physical model will therefore suffice. Figures 1 and 2 depict the geometrical arrangement and grid-point specification for this model. The imposed motion of the piston in the cylinder is sinusoidal. The valves are assumed to be unobstructed ports which open and close instantaneously at the proper times during the four-stroke cycle. Ten z-planes (see Figure 2) were used in the work reported here.

2. Governing Equations and Numerical Method

Since the present work deals mainly with the difficulties inherent in the numerical analysis of three-dimensional flowfields in cylindrical coordinates, the working fluid (air) is treated for simplicity as a single component calorically-perfect gas. An inviscid fluid is assumed, but it is emphasized that the method discussed here in no way requires such an assumption. For this case, the non-dimensional governing equations in cylindrical component form are

$$\frac{\partial u}{\partial t} = \frac{v^2}{r} - \left(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z}\right) - \frac{T}{\gamma p}\frac{\partial p}{\partial r}$$

$$\frac{\partial v}{\partial t} = -\frac{uv}{r} - \left(u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z}\right) - \frac{T}{\gamma p}\frac{1}{r}\frac{\partial p}{\partial \phi}$$

$$\frac{\partial w}{\partial t} = -\left(u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z}\right) - \frac{T}{\gamma p}\frac{\partial p}{\partial z} \tag{3}$$

$$\frac{\partial p}{\partial t} = -\left(u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \phi} + w\frac{\partial p}{\partial z}\right) - \gamma p\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z}\right) \tag{4}$$

$$\frac{\partial T}{\partial t} = -\left(u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \phi} + w\frac{\partial T}{\partial z}\right) - (\gamma - 1)\,T\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z}\right) \tag{5}$$

Equations (1) to (3) are the r, ϕ, and z component equations for conservation of momentum. Equations (4) and (5) are the equations for mass and energy conservation. The factor of γ appears in the non-dimensional forms because of the choice of reference velocity,

$$V_r = (\gamma R T_r)^{1/2} \tag{6}$$

for the system. Note that (u, v, w) are the (r, ϕ, z) velocity components, and that density has been eliminated in favor of temperature and pressure with the equation of state,

$$p = \rho R T \tag{7}$$

in dimensional variables.

The numerical algorithm used in this work is the second-order, explicit, time-dependent finite-difference scheme due to MacCormack [6]. Again, however, the technique given here for the treatment of coordinate singularities is not dependent on the use of a particular algorithm. The integration step size, or time step, is bounded by the Courant-Friedrichs-Lewy (CFL) condition,

$$\Delta t = \frac{\Delta}{|\mathbf{V}| + a} \tag{8}$$

where

$$\begin{aligned} \Delta &= \text{local grid spacing} \\ |\mathbf{V}| &= \text{magnitude of local flow velocity} \\ a &= \text{local speed of sound} \end{aligned}$$

Equation (8) was applied on each time step at all points in the flowfield and some multiple (usually .7) of the smallest value obtained used as the actual step size.

Attempts to directly implement the numerical algorithm for equations (1) to (5) will encounter difficulty due to the indeterminacy of ϕ when $r = 0$. It is emphasized that this is due purely to the representation of the governing equations in a particular coordinate system. The flowfield itself is in no way singular along the centerline. As an example, the flow of a uniform stream of speed U in the $+x$ direction is given by

$$\mathbf{V} = U\hat{\imath} \tag{9}$$

at an arbitrary field point in Cartesian coordinates. At the same field point in cylindrical coordinates the representation is

$$\mathbf{V} = \hat{r}U\cos\phi - \hat{\phi}U\sin\phi = u(r,\phi)\,\hat{r} + v(r,\phi)\,\hat{\phi} \tag{10}$$

which is clearly a function of the azimuthal coordinate ϕ. This flow will have constant properties along any radial line from the origin, but *at* the origin, where the radial lines meet, the representation in cylindrical coordinates is necessarily multivalued.

The technique described here for resolving the difficulty consists of applying L'Hospital's Rule for indeterminate forms to equations (1) to (5). Although no formal argument is made to show that the cylindrical coordinate equations are actually indeterminate forms at $r = 0$, justification for this can be given on physical grounds. If the terms containing $(1/r)$ do not combine to produce an indeterminate form, then the time derivatives of the flowfield quantities at the centerline will be infinite, a situation which is not allowed on physical grounds.

Formal application of L'Hospital's Rule to equations (1) to (5) yields:

$$\frac{\partial u}{\partial t} = \lim_{r=0}\left(2v\frac{\partial v}{\partial r}\right) - u\frac{\partial u}{\partial r} - \lim_{r=0}\left(v\frac{\partial^2 u}{\partial r\,\partial\phi} + \frac{\partial v}{\partial r}\frac{\partial u}{\partial\phi}\right) - w\frac{\partial u}{\partial z} - \frac{T}{\gamma p}\frac{\partial p}{\partial r} \tag{11}$$

$$\frac{\partial v}{\partial t} = -\lim_{r=0}\left(u\frac{\partial v}{\partial r} + v\frac{\partial u}{\partial r}\right) - u\frac{\partial v}{\partial r} - \lim_{r=0}\left(v\frac{\partial^2 v}{\partial r\,\partial\phi} + \frac{\partial v}{\partial r}\frac{\partial v}{\partial\phi}\right)$$

$$- w\frac{\partial v}{\partial z} - \frac{T}{\gamma p}\lim_{r=0}\frac{\partial^2 p}{\partial r\,\partial\phi} \tag{12}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial r} - \lim_{r=0}\left(v\frac{\partial^2 w}{\partial r\,\partial\phi} + \frac{\partial v}{\partial r}\frac{\partial w}{\partial\phi}\right) - w\frac{\partial w}{\partial z} - \frac{T}{\gamma p}\frac{\partial p}{\partial z}$$

$$\frac{\partial p}{\partial t} = -u\frac{\partial p}{\partial r} - \lim_{r=0}\left(v\frac{\partial^2 p}{\partial r\,\partial\phi} + \frac{\partial v}{\partial r}\frac{\partial p}{\partial\phi}\right) - w\frac{\partial p}{\partial z}$$

$$- \gamma p\left[\lim_{r=0}\left(2\frac{\partial u}{\partial r} + \frac{\partial^2 v}{\partial r\,\partial\phi}\right) + \frac{\partial w}{\partial z}\right] \tag{14}$$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial r} - \lim_{r=0}\left(v\frac{\partial^2 T}{\partial r\,\partial\phi} + \frac{\partial v}{\partial r}\frac{\partial T}{\partial\phi}\right) - w\frac{\partial T}{\partial z}$$

$$- (\gamma - 1)\,T\left[\lim_{r=0}\left(2\frac{\partial u}{\partial r} + \frac{\partial^2 v}{\partial r\,\partial\phi}\right) + \frac{\partial w}{\partial z}\right] \tag{15}$$

Considerable time was spent attempting to evaluate the limits found in the above equations; i.e., attempting to identify fixed limits for terms like $\partial^2 p/\partial r\partial\phi\,|_{r=0}$ that would apply in the general case. No arguments could be constructed to show what such limits should be. The following approach was then adopted:

1. There probably are no specific limits that should occur; the required terms may be both time and azimuth dependent.

2. It is not necessary to know *a priori* what the limiting terms in the L'Hospital equations should be. That is, there is no way in a finite difference solution to obtain any information about any quantity more accurately than is inherent in the initially chosen grid resolution. Therefore, such terms as $\lim_{r=0} \partial^2 p/\partial r\partial\phi$ can be correctly treated by replacing the limit expression with the best available finite difference. The required limits are then resolved to the accuracy of the grid spacing.

These arguments allow a set of equations to be developed that seem to be consistent with the finite-difference approximations inherent in the overall scheme, yet are free of coordinate singularities. There is, however, another objection to the use of L'Hospital's Rule in cases where the ϕ-derivatives are not identically zero. In such cases centerline computations are performed at neighboring "points" in ϕ-space which are actually the same point (somewhere on the z-axis) in physical space. Since the arc length $r\,\Delta\phi$ goes to zero at the centerline, it would appear that equation (8) would require a limiting time step of zero to guarantee stability, thus preventing the calculation from being advanced in time. However, despite these reservations, it was decided to implement equations (11) to (15) within the overall MacCormack algorithm. It was thought that the z-axis might be special in some unanticipated way that would invalidate the apparent time-step limitation. This has been the case; the reason is not known.

A further problem is that the region $r < 0$ does not exist, hence only forward differences can be used in the radial direction at the centerline. This was shown to cause some difficulty which was eliminated by using a second-order one-sided first-difference formula at the centerline in the radial direction.

4. Computational Method using L'Hospital's Rule

The computations for the "ring" of points at $r = 0$ are handled much the same as those at any other place in the flowfield, with the xception that different governing equations are used after the spatial differences are obtained, and a few extra cross-derivative terms are required. Prior to spatial differencing some conditions are imposed on the "ring" at $r = 0$, based on the physics of the flow at the centerline. The first condition is obtained by noticing that the multivalued nature of the flowfield does not extend to pressure, temperature, and the z-component of velocity. Thus, prior to spatial differencing these values are averaged on the "ring" of constant z and varying ϕ at $r = 0$. The entire ring is then reset to the average value. The second condition derives from the fact that the true velocity vector through any point can have only one direction in physical space, and at the centerline lies in the (x, z) plane, since in our problem this is a symmetry plane through which there can be no mass flow. If the velocity vector at a given value of z has the value U_{avg}, then the u and v component values in cylindrical coordinates must be

$$\lim_{r \to 0} u(r, \phi, z) = U_{avg}(z) \cos \phi \tag{16}$$

$$\lim_{r \to 0} v(r, \phi, z) = -U_{avg}(z) \sin \phi \tag{17}$$

It is emphasized that (16) and (17) hold exactly for the velocity distribution on the "ring". To determine U_{avg}, one computes the average of all x-components of velocity on the "ring" using

$$U(\phi, z) = u(r, \phi, z) \cos \phi - v(r, \phi, z) \sin \phi \tag{18}$$

to yield each x-component. This completes the special procedures used at the centerline.

Note that use of the method, while made more convenient by the presence of a plane of symmetry, is not restricted to cases where one exists. Even if the flow as a whole has no plane of symmetry, it is still true that the velocity "field" at the centerline must have the form of equations (16) and (17), with the exception that "ϕ" becomes "$\phi - \phi_0$", where ϕ_0 denotes the angle of the velocity vector (now no longer 0 or π) in a given z-plane. The velocity field on the "ring" will be symmetrical about a line at angle ϕ_0 in that z-plane, with ϕ_0 different on each plane. The value of ϕ_0 is found from

$$V(\phi, z) = u(r, \phi, z) \sin \phi + v(r, \phi, z) \cos \phi \tag{19}$$

which gives the resultant velocity in the y-direction, and

$$\phi_0 = \tan^{-1}(V/U) \tag{20}$$

Equation (20) is valid for each choice of ϕ yielding values of U and V from equations (18) and (19). It seems best to average the result obtained from the entire "ring" at $r = 0$.

Since there is no region defined by $r < 0$, rearward radial differences cannot be obtained at the centerline. This results in first-order accuracy at the centerline when the normal MacCormack algorithm is applied. It was found necessary to alter the standard differencing scheme of Reference [6] by using the second-order one-sided first-difference approximation

$$\left.\frac{\partial g}{\partial r}\right|_{r=0} = \frac{-3g(r) + 4g(r + \Delta r) - g(r + 2\,\Delta r)}{2\,\Delta r} + O(\Delta r)^3 \tag{21}$$

for radial differencing at the centerline.

6. Results

The main result of this paper is that the above-described finite-difference technique does in fact allow the stable computation of flowfield values at the centerline in cylindrical coordinates for non-axisymmetric problems. There are also quantitative results which show that the method produces accurate solutions. The accuracy can be verified by considering the indicator diagram of Figure 3 for a 3000 RPM engine

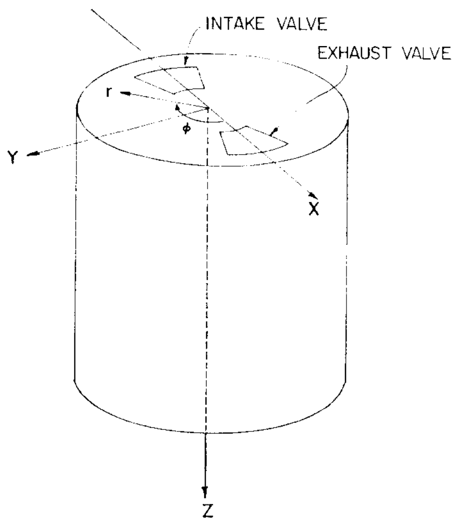

FIG. 1. Cylindrical engine model and coordinate system.

test using the geometry of Figs. 1. and 2. The governing equations implemented in this work are isentropic in nature, hence the computed results should follow an isentropic curve. The indicator diagram of Fig. 3 plots the pressure vs. specific volume ($1/\rho$) history at an arbitrary point in the flowfield throughout the four-stroke calculation. The solid line is the correct isentrope for the compression stroke (except

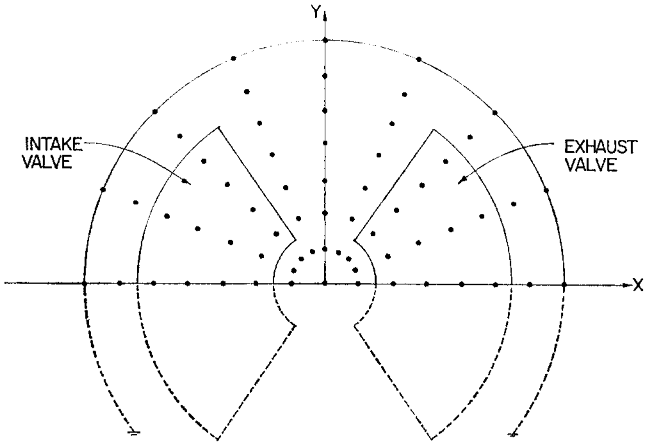

FIG. 2. Top of cylinder from inside including grid layout.

during the heat addition phase of the computation, when the internal fluid is shifted to a different isentrope), and the dots and circles show calculated values. It is seen that the computed results fall exactly on the isentrope during the compression stroke.

An examination of the behavior of the computed results during the power stroke (following heat addition) demonstrates the need for second-order spatial accuracy in the centerline calculations. After heat-addition at the top of the compression stroke, the computed results should follow the isentrope associated with the new conditions throughout the rest of the cycle. However, it is seen that the calculation using first-order differencing (solid dots) at the centerline suffers a severe increase in entropy (loss of total pressure relative to the correct isentrope) before converging to a new isentrope for the remainder of the power stroke. When second-order accurate differencing in the radial direction at the centerline is used, the computed results (open circles) continue to follow the isentrope established after heating.

The reason for the loss of total pressure is that, as shown in Fig. 4, severe and physically unrealistic pressure gradients are established shortly after heat addition when first-order accurate differencing is used. The strong gradients produce an entropy increase through the action of an artificial viscosity term that is required to stabilize the calculation ([2], [3]). The requirement for an explicit artificial viscosity term exists for 3-D solutions in cartesian coordinates also, hence is totally unrelated to the coordinate-singularity problem. The artificial viscosity is of the same order as the truncation error in the method, and produces an effect only in the presence of strong gradients such as those set up by the unstable first-order accurate centerline calculation.

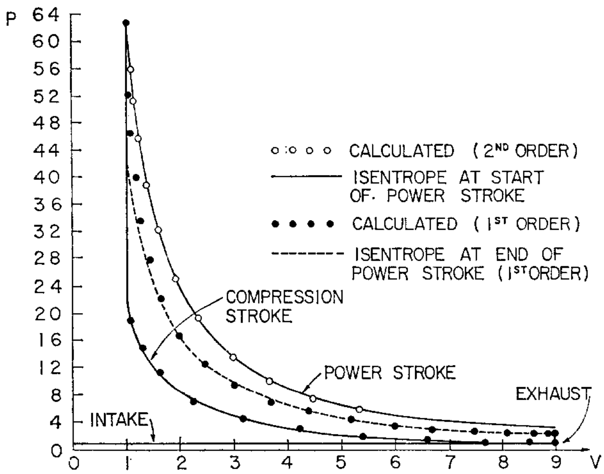

FIG. 3. Indicator diagram for cylindrical engine for 3000 RPM inviscid calculation vs isentropic law.

When second-order differencing at the z-axis was implemented, the heat addition process failed to produce the strong gradients of Fig. 4. Instead, the pressure at a given time remained spatially constant within the cylinder to better than three significant digits, a result in agreement with the corresponding test case in cartesian coordinates.

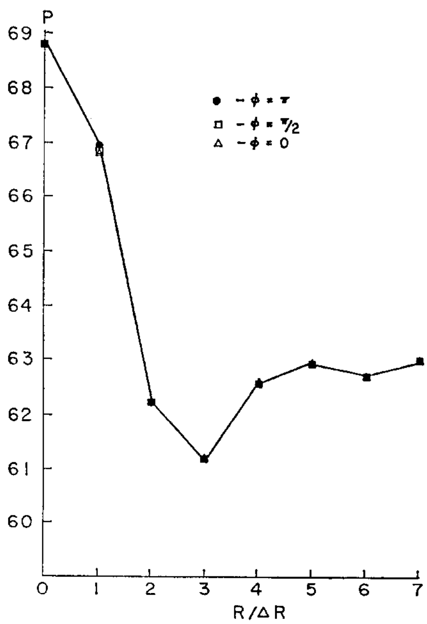

FIG. 4. Pressure distribution in radial direction with first-order differencing for crank angle of 360°, at $z = 3\Delta z$.

7. Conclusions

1. This paper has detailed a technique for calculating non-axisymmetric flows in cylindrical coordinates, a method believed to be the first of its kind for finite-difference methods. It has been demonstrated for a problem of current interest, and has been shown to provide verifiably exact results in certain cases.

2. The technique has been applied to a problem whose physics dictate the existence of a plane of symmetry. However, it is shown that the use of the method does not depend on the existence of such a symmetry plane, and the appropriate computational technique is specified for the case where no plane of symmetry exists.

3. The numerical approach developed here is applied within the context of a second-order, explicit, time-dependent finite-difference scheme of MacCormack [6]; however, there is nothing in the method which would seem to restrict it to a particular finite-difference scheme.

4. Second-order accuracy in the spatial differencing at the centerline seems to be a requirement for the method. This is in contrast to other regions of the flowfield (such as the wall) where one-sided, first-order accurate differencing invariably suffices to yield a reasonable solution. The accuracy requirement is easily met through the use of a three-term, one-sided first-difference approximation for radial derivatives.

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References

1. M. D. Griffin, J. D. Anderson, Jr., and R. Diwakar, "Navier–Stokes solutions for the flowfield in an internal combustion engine," *AIAA J.* **14** (1976), 1665–1666.
2. M. D. Griffin, "Numerical Solutions for Two- and Three-Dimensional Non-Reacting Flowfields in an Internal Combustion Engine," Ph.D. Thesis, Department of Aerospace Engineering, University of Maryland, 1977.
3. M. D. Griffin, R. Diwakar, J. D. Anderson, Jr., and E. Jones, "Computational Fluid Dynamics Applied to the Flow in an Internal Combustion Engine," AIAA Paper No. 78-57, 16th AIAA Aerospace Sciences Meeting, Huntsville, Alabama, January 16–18, 1978.
4. R. Diwakar, J. D. Anderson, Jr., M. D. Griffin, and E. Jones, Inviscid solutions for the flowfield in an internal combustion engine, *AIAA J.* **14** (1976), 1667–1668.
5. R. Diwakar, "Interaction of Combustion with the Aerodynamic Flowfield in an Internal Combustion Reciprocating Engine—A Two-Dimensional Solution," Ph.D. Thesis, Department of Aerospace Engineering, University of Maryland, 1977.
6. R. W. MacCormack, "The Effect of Viscosity on Hypervelocity Impact Cratering," AIAA Paper No. 69-354, AIAA Hypervelocity Impact Conference, Cincinatti, Ohio, April 30–May 2, 1969.